Spherical two-distance sets & spectral theory of signed graphs

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arxiv: 1708.02317

arxiv: 1907.12466.

arxiv: 2006.06633

Spherical two-distance set $= \{ v_1, v_2, \cdots, v_N \in \mathbb{R}^d :$

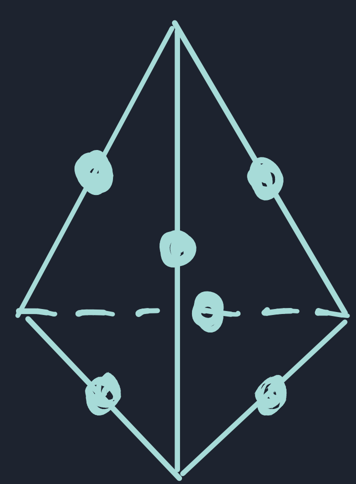
$\forall i \ |v_i| = 1$ & $\forall i \neq j \ \langle v_i, v_j \rangle$ takes only 2 values $\}$.

<u>Question</u> Find max size of spherical 2-distance set in $\mathbb{R}^d$.

$$\binom{d+1}{2} = \frac{1}{2} d(d+1) \leq N(d) \leq \frac{1}{2} d(d+3)$$

Delsarte, Goethals, Seidel

<u>Example</u>



midpoints of regular simplex.

$\mathbb{R}^{d+1}$: $e_1, \cdots, e_{d+1}$ $\left\langle \frac{e_i + e_j}{2}, \frac{e_k + e_l}{2} \right\rangle$.

[Glazyrin, Yu] $N(d) = \frac{1}{2} d(d+1)$ $\forall d \geq 7$ but $d \neq 5^2-3, 7^2-3, 9^2-3, \cdots$

Question What if inner products are fixed?

Given $-1 \le \beta < \alpha < 1$.

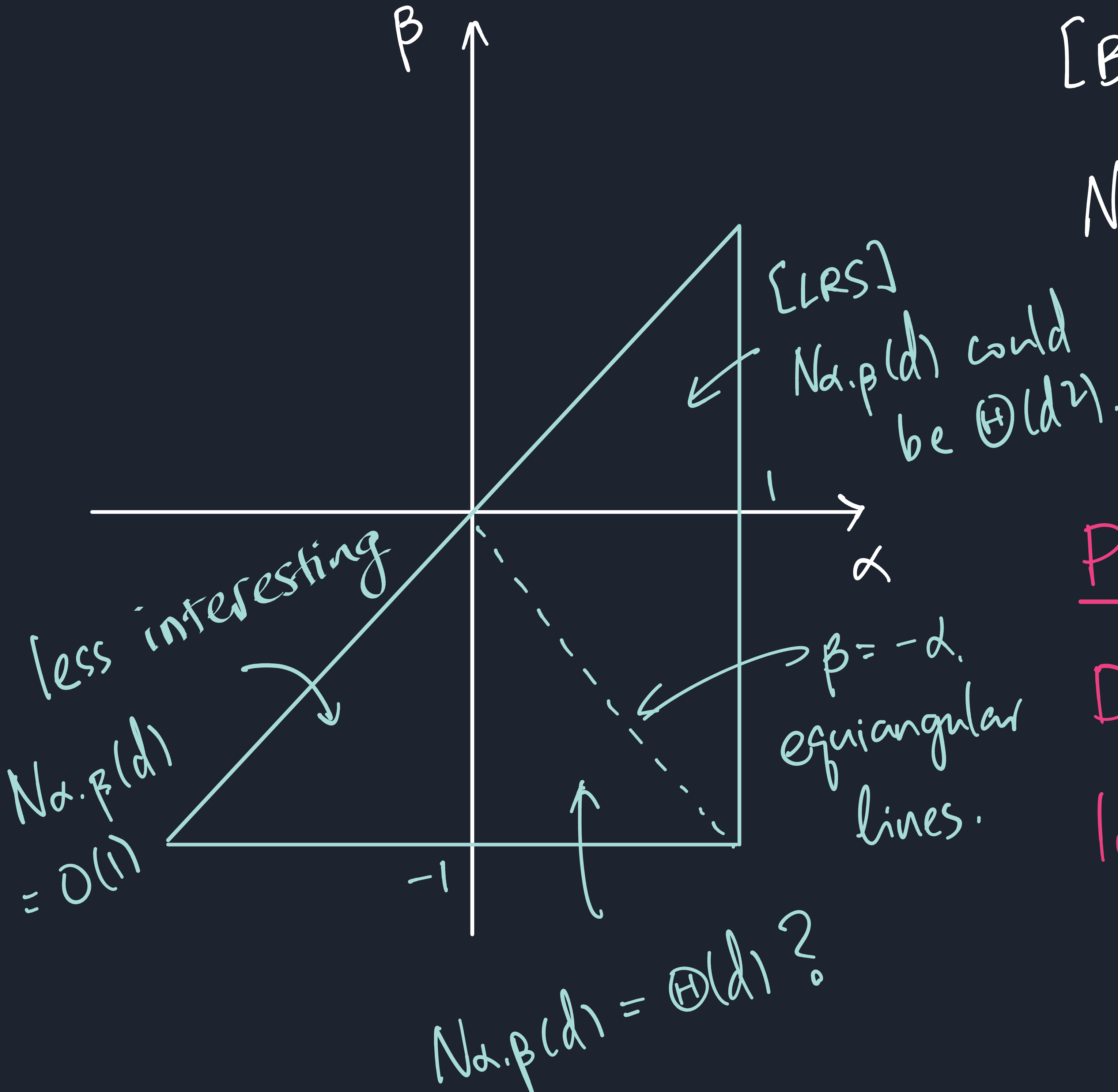
$N_{\alpha,\beta}(d)$ = max size of set of unit vectors $v_1, \cdots, v_N$ in $\mathbb{R}^d$ s.t. $\forall\, i \neq j$ $\langle v_i, v_j \rangle \in \{\alpha, \beta\}$.

[Neumair] $N_{\alpha,\beta}(d) \le 2d+1$ unless $1-\alpha / \alpha-\beta \in \mathbb{Z}$

[Larman, Rogers, Seidel]

$$N_{\alpha,\beta}(d) = \Theta(d^2) \text{ if } 0 \le \beta < \alpha < 1 \text{ and } \frac{1-\alpha}{\alpha-\beta} \in \mathbb{Z}$$

Observation $N_{\alpha,\beta}(d) \le 1 - 1/\alpha$ if $0 \le \beta < \alpha < 0$.



[Balla, Dräxler, Keevash, Sudakov]

$$N_{\alpha,\beta}(d) \le 2\left(1-\frac{\alpha}{\beta}\right)d + o(d).$$

if $-1 \le \beta < 0 \le \alpha < 1$.

<u>Problem</u> Fix $-1 \le \beta < 0 \le \alpha < 1$.

Determine $N_{\alpha,\beta}(d)$ for large d

In particular, find

$$\lim_{d\to\infty} \frac{N_{\alpha,\beta}(d)}{d}$$

Equiangular line $\beta = -\alpha$

$$\lambda = \frac{1-\alpha}{\alpha-\beta} = \frac{1-\alpha}{2\alpha}$$

DEF [J-Polyanskii 19] Spectral radius order

$k(\lambda) :=$ smallest k s.t. $\exists$ k-vertex G whose adjacency matrix has largest eigenvalue $\lambda_1(G) = \lambda$.

(Set $k(\lambda) = \infty$ if no such G exists)

Long history:

Lemmens
- Seidel 73.

⋮

Bukh 16.

Balla-Draxler-Sudakov-Keevash 18

THM [JTYYZ] Fix $\alpha > 0$. set $\lambda = \frac{1-\alpha}{2\alpha}$ for $d \geq d_0(\alpha)$.

$$N_{\alpha,-\alpha}(d) = \begin{cases} \left\lfloor \frac{k(\lambda)(d-1)}{k(\lambda)-1} \right\rfloor & \text{if } k(\lambda) < \infty \\ d + o(d) & \text{if } k(\lambda) = \infty \end{cases}$$

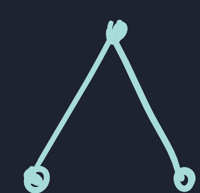
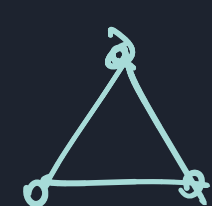
<u>DEF</u> [J-Polyanskii 19] Spectral radius order

$$k(\lambda) := \text{smallest } k \text{ s.t. } \exists\ k\text{-vertex } G \text{ whose } \lambda_1(G) = \lambda.$$

<u>THM</u> [JTYYZ] Fix $\alpha > 0$. set $\lambda = \frac{1-\alpha}{2\alpha}$ for $d \geq d_0(\alpha)$.

$$N_{\alpha, -\alpha}(d) = \begin{cases} \left\lfloor \frac{k(\lambda)(d-1)}{k(\lambda)-1} \right\rfloor & \text{if } k(\lambda) < \infty \\ d + o(d) & \text{if } k(\lambda) = \infty \end{cases}$$

<u>Example</u>

α	λ	G	$k(\lambda)$	$N_{\alpha,-\alpha}(d)$
$1/3$	1		2	$2d$
$1/(1+2\sqrt{2})$	$\sqrt{2}$		3	$\frac{3d}{2}.$
$1/5.$	$2.$		3	$\frac{3d}{2}.$

Connection between $N_{\alpha,\dots\alpha}(d)$ and $k(\lambda)$

$k(\lambda) :=$ smallest k s.t. $\exists$ k-vertex G whose $\lambda_1(G) = \lambda$.

Equiangular lines in $\mathbb{R}^d$

$V = \{v_1, \dots, v_N\} \subseteq \mathbb{R}^d$.

$|v_i| = 1$. $\quad \langle v_i, v_j \rangle = \pm\alpha$

$$\begin{cases} \text{Gram matrix } (\langle v_i, v_j \rangle)_{ij} \succeq 0 \\ \text{rank}(\text{Gram matrix}) \le d. \end{cases}$$

N-vertex graph G.

$V = \{v_1, \dots, v_N\}$

$v_i \sim v_j$ if $\langle v_i, v_j \rangle = -\alpha$

A_G - adjacency matrix of G.

$$\begin{cases} \underbrace{\lambda}_{\frac{1-\alpha}{2\alpha}} I - A_G + \frac{1}{2} J \succeq 0 \quad \text{(PSD)} \\ \text{rank}(\lambda I - A_G + \frac{1}{2} J) \le d. \quad \text{(RANK)} \end{cases}$$

<u>Goal</u> Given d and λ. find largest N-vertex graph with …

(PSD) + (RANK).

Goal Given d and $\lambda \left(= \frac{1-d}{2\alpha}\right)$. find largest N-vertex graph G s.t.

(PSD) $\lambda I - A_G + \frac{1}{2} J \succeq 0$ & (RANK) $\operatorname{rank}\left(\lambda I - A_G + \frac{1}{2} J\right) \leq d$

Construction Take k-vertex graph G_0 with $\lambda_1(G_0) = \lambda$.

Take G = disjoint $\underline{\ell}$ copies of G_0.

$$\underbrace{\{\lambda, \lambda, \dots, \lambda}_{\ell}, \dots \}$$

(PSD): $\underbrace{\lambda I - A_G}_{\succeq 0} + \underbrace{\frac{1}{2} J}_{\succeq 0} \succeq 0$

(RANK): $\operatorname{rank}\left(\lambda I - A_G + \frac{1}{2} J\right) \leq \operatorname{rank}(\lambda I - A_G) + 1 = \ell(k-1) + 1 \underline{\leq d}.$

WANT

PROP $N_{\alpha,-\alpha}(d) \geq \dfrac{k(\lambda)\, d}{k(\lambda) - 1} + O(1).$ $\quad \ell \approx \dfrac{d}{k-1}$ $\quad |G| = \dfrac{kd}{k-1}$

THM [JTYYZ] The reversed inequality holds

Problem Fix $-1 \le \beta < 0 \le \alpha < 1$. Determine $N_{\alpha,\beta}(d)$ for large d

In particular, find $\lim\limits_{d\to\infty} \dfrac{N_{\alpha,\beta}(d)}{d}$

Answer When $\beta = -\alpha$, $\lim = k(\lambda) / k(\lambda) - 1$, where $\lambda = \dfrac{1-\alpha}{2\alpha}$

spectral radius order.

Generalize $k(\lambda)$ for general α and β.

Spherical 2-distance set

$\{v_1, \cdots, v_N\} \subseteq \mathbb{R}^d$.

$|v_i| = 1$. & $\langle v_i, v_j \rangle = \alpha$ or β.

Gram $\succeq 0$ rank(Gram) $\le d$.

N-vertex graph G

$\{v_1, \cdots, v_N\}$. $v_i \sim v_j$ iff $\langle v_i, v_j \rangle = \beta$

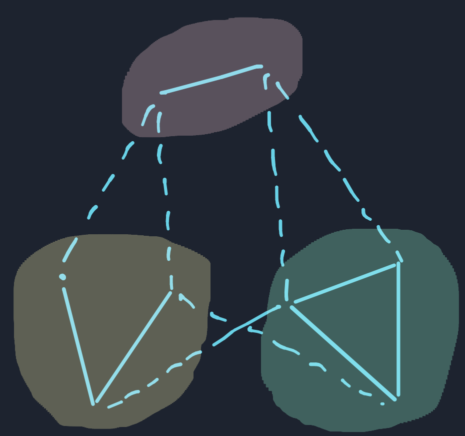
(PSD): $\underset{\underset{\frac{1-\alpha}{\alpha-\beta}}{\|}}{\lambda} I - A_G + \underset{\underset{\frac{\alpha}{\alpha-\beta}}{\uparrow}}{\mu} J \succeq 0$

(RANK): $\operatorname{rank}(\lambda I - A_G + \mu J) \le d$.

Goal: Given λ, μ, d, find largest N-vtx graph G with (PSD) & (RANK).

Goal Given λ, μ, d find largest N-vertex G s.t. $\lambda I - A_G + \mu J \succeq 0$ and $\operatorname{rank}(\lambda I - A_G + \mu J) \le d$.

DEF Signed graph $G^{\pm}$.



Adjacency matrix $\begin{bmatrix} 0 & \cdot & \pm 1 \\ \pm 1 & \ddots & 0 \\ 0 & & 0 \end{bmatrix}$, eigenvalues,

Valid t-coloring, chromatic number $\chi(G^{\pm})$ and $\lambda_1(G^{\pm}) = \lambda$.

Construction Take a signed graph $G^{\pm}$ s.t. $\chi(G^{\pm}) \le \lfloor \frac{\alpha}{-\beta} \rfloor + 1 := p$.

Let $V_1, V_2, \dots, V_t$ be valid t-coloring. $(t := \chi(G^{\pm}) \le p)$.

Let G be s.t. $A_G = A_{G^{\pm}} + A_{K_{V_1, V_2, \dots, V_t}}$.

(PSD): $\lambda I - A_{G^{\pm}} - A_{K_{V_1, V_2, \dots, V_t}} + \mu J \succeq 0$.

WANT. $\le d$.

(RANK): $\operatorname{rank}(\lambda I - A_G + \mu J) \approx \operatorname{rank}(\lambda I - A_{G^{\pm}}) = |G^{\pm}| - \operatorname{mult}(\lambda, G^{\pm})$

Generalize $k(\alpha)$ for general α, β.

$$\lambda = \frac{1-\alpha}{\alpha-\beta}$$

$$\mu = \frac{\alpha}{\alpha-\beta}$$

$$p = \left\lfloor \frac{\alpha}{-\beta} \right\rfloor + 1$$

Def $$k_p(\lambda) = \inf\left\{ \frac{|G^\pm|}{\mathrm{mult}(\lambda, G^\pm)} : \begin{array}{l} \lambda_1(G^\pm) = \lambda \\ \chi(G^\pm) \le p \end{array} \right\}.$$

PROP $$N_{\alpha,\beta}(d) \geq \frac{k_p(\lambda)\, d}{k_p(\lambda) - 1} + o(d).$$

CONJ The reverse inequality holds

Examples

α	β	λ	p	$k_p(\lambda)$	$N_{\alpha,\beta}(d)$
α	$-\alpha$	$\frac{1-\alpha}{2\alpha}$	$2.$	$k(\lambda)$	$\frac{k(\lambda)}{k(\lambda)-1} d.$
$\alpha + 2\beta < 0$		$\frac{1-\alpha}{\alpha-\beta}$	$\le 2.$	$k(\lambda)$	$\frac{k(\lambda)}{k(\lambda)-1} d.$
		1	$\geq 3.$	$p.$	$\frac{p}{p-1} d.$
		$\sqrt{2}$	≥ 3	2	$2\,d$
		$\sqrt{3}$	3	$7/3$	$7/4\, d$
		$\sqrt{3}$	≥ 4	2	$2d.$