Spherical two-distance sets & spectral theory of sign graphs

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Joint work with

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arxiv: 1708.02317

arxiv: 1907.12466.

arxiv: 2006.06633

Spherical two-distance set $= \{ v_1, v_2, \cdots, v_N \in \mathbb{R}^d :$

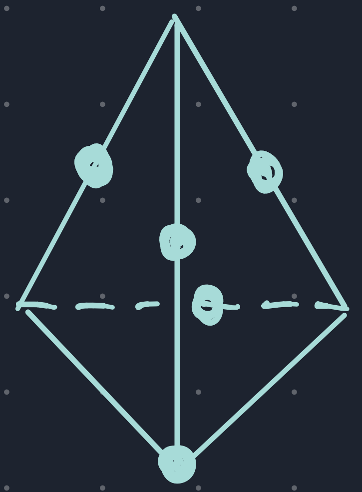
$\forall i \ |v_i| = 1$ & $\forall i \neq j \ \langle v_i, v_j \rangle$ takes only 2 values $\}$.

Question Find max size of spherical 2-distance set in $\mathbb{R}^d$.

$$\binom{d+1}{2} = \frac{1}{2} d(d+1) \leq N(d) \leq \frac{1}{2} d(d+3)$$

Delsarte, Goethals, Seidel

Example



midpoints of regular simplex.

[Glazyrin, Yu] $N(d) = \frac{1}{2} d(d+1) \quad \forall d \geq 7$ but $d \neq \text{odd}^2 - 3$.

Question What if inner products are fixed?

Given $-1 \le \beta < \alpha < 1$.

$N_{\alpha,\beta}(d)$ = max size of set of unit vectors $v_1, \cdots, v_N$ in $\mathbb{R}^d$ s.t. $\forall\, i \neq j$ $\langle v_i, v_j \rangle \in \{\alpha, \beta\}$.

[Neumair] $N_{\alpha,\beta}(d) \le 2d+1$ unless $1-\alpha / \alpha-\beta \in \mathbb{Z}$

[Larman, Rogers, Seidel]

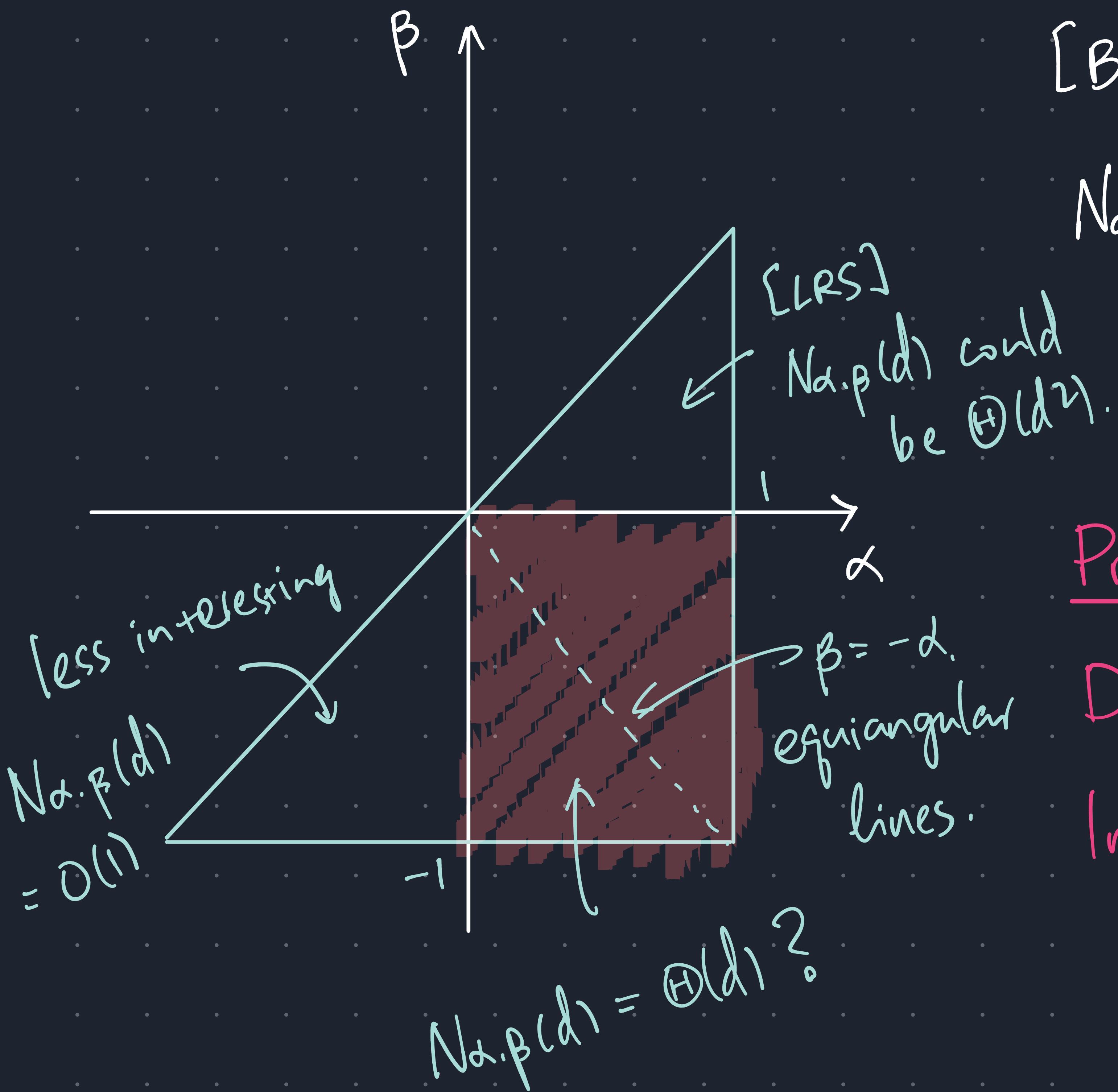
$N_{\alpha,\beta}(d) = \Theta(d^2)$ if $0 \le \beta < \alpha < 1$ and $\dfrac{1-\alpha}{\alpha-\beta} \in \mathbb{Z}$

Observation $N_{\alpha,\beta}(d) \le 1 - 1/\alpha$ if $0 \le \beta < \alpha < 0$.

[Balla, Dräxler, Keevash, Sudakov]

$$N_{\alpha,\beta}(d) \le 2\left(1 - \frac{\alpha}{\beta}\right) d + o(d).$$

if $-1 \le \beta < 0 \le \alpha < 1$.



Problem Fix $-1 \le \beta < 0 \le \alpha < 1$.

Determine $N_{\alpha,\beta}(d)$ for large d

In particular, find

$$\lim_{d \to \infty} \frac{N_{\alpha,\beta}(d)}{d}$$

Equiangular line $\beta = -\alpha$

$$\lambda = \frac{1-\alpha}{\alpha-\beta} = \frac{1-\alpha}{2\alpha}$$

DEF [J-Polyanskii 19] Spectral radius order

$k(\lambda) :=$ smallest k s.t. $\exists$ k-vertex G whose adjacency matrix has largest eigenvalue $\lambda_1(G) = \lambda$.

(Set $k(\lambda) = \infty$ if no such G exists)

THM [JTYYZ] Fix $\alpha > 0$. set $\lambda = \frac{1-\alpha}{2\alpha}$ for $d \geq d_0(\alpha)$ $\quad 2^{O(\frac{1}{\alpha^2})}$

$$N_{\alpha,-\alpha}(d) = \begin{cases} \left\lfloor \frac{k(\lambda)(d-1)}{k(\lambda)-1} \right\rfloor & \text{if } k(\lambda) < \infty \\ d + o(d) & \text{if } k(\lambda) = \infty \end{cases}$$

DEF [J-Polyanskii 19] Spectral radius order

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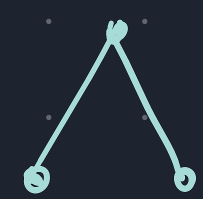
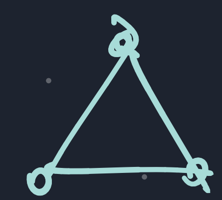
Example

α	λ	G	$k(\lambda)$	$N_{\alpha,-\alpha}(d)$
$1/3$	1		2	$2d$
$1/(1+2\sqrt{2})$	$\sqrt{2}$		3	$\frac{3d}{2}$.
$1/5$.	2.		3	$\frac{3d}{2}$.

Connection between $N_{\alpha,\ldots\alpha}(d)$ and $k(\lambda)$

$k(\lambda) :=$ smallest k s.t. $\exists$ k-vertex G whose $\lambda_1(G) = \lambda$.

Equiangular lines in $\mathbb{R}^d$	N-vertex graph G.
$V = \{v_1, \ldots, v_N\} \subseteq \mathbb{R}^d$.	$V = \{v_1, \cdots, v_N\}$
$\lvert v_i \rvert = 1$. $\quad \langle v_i, v_j \rangle = \pm\alpha$	$v_i \sim v_j$ if $\langle v_i, v_j \rangle = -\alpha$
	A_G - adjacency matrix of G.
$\begin{cases} \text{Gram matrix } (\langle v_i, v_j \rangle)_{ij} \succeq 0 \\ \text{rank}(\text{Gram matrix}) \leq d. \end{cases}$	$\begin{cases} \underline{\lambda} I - A_G + \frac{1}{2} J \succeq 0 \quad \text{(PSD)} \\ \uparrow \ \frac{1-\alpha}{2\alpha} \\ \text{rank}(\lambda I - A_G + \frac{1}{2} J) \leq d. \quad \text{(RANK)} \end{cases}$

$\underline{\text{Goal}}$ Given $\underline{\underline{d}}$ and $\underline{\underline{\lambda}}$, find largest N-vertex graph with ...

(PSD) + (RANK).

Goal Given d and $\lambda \left(= \frac{1-\alpha}{2\alpha}\right)$. find largest N-vertex graph G s.t.

(PSD) $\lambda I - A_G + \frac{1}{2}J \succeq 0$ & (RANK) $\operatorname{rank}\left(\lambda I - A_G + \frac{1}{2}J\right) \leq d$

Construction Fink $\underline{k}$-vertex connected graph G_0 with $\lambda_1(G_0) = \lambda$.

Take G = disjoint $\underline{\ell}$ copies of G_0.

$$\{\underbrace{\lambda, \lambda, \cdots, \lambda}_{\ell}, \cdots\}$$

(PSD): $\underbrace{\lambda I - A_G}_{\succeq 0} + \underbrace{\tfrac{1}{2}J}_{\succeq 0} \succeq 0$

(RANK): $\operatorname{rank}\left(\lambda I - A_G + \frac{1}{2}J\right) \leq \operatorname{rank}(\lambda I - A_G) + 1 = \ell(k-1) + 1 \underset{\text{WANT}}{\leq} d.$

PROP $N_{\alpha,-\alpha}(d) \geq \dfrac{k(\lambda)d}{k(\lambda)-1} + O(1).$ $\qquad \ell \approx \dfrac{d}{k-1} \qquad |G| = \dfrac{kd}{k-1}$

THM [JTYY2] The reversed inequality holds

Problem Fix $-1 \le \beta < 0 \le \alpha < 1$. Determine $N_{\alpha,\beta}(d)$ for large d

In particular, find $\lim_{d\to\infty} \frac{N_{\alpha,\beta}(d)}{d}$

spectral radius order.

Answer When $\beta = -\alpha$, $\lim = k(\lambda) / k(\lambda) - 1$, where $\lambda = \frac{1-\alpha}{2\alpha}$

Question Generalize $k(\lambda)$ for $N_{\alpha,\beta}(d)$.

Spherical 2-distance set	N-vertex graph G
$\{v_1, \dots, v_N\} \subseteq \mathbb{R}^d$	$\{v_1, \dots, v_N\}$. $v_i \sim v_j$ iff $\langle v_i, v_j \rangle = \beta$
$\lvert v_i \rvert = 1$ & $\langle v_i, v_j \rangle = \alpha$ or β.	(PSD): $\underbrace{\lambda}_{= \frac{1-\alpha}{\alpha-\beta}} I - A_G + \underbrace{\mu}_{= \frac{\alpha}{\alpha-\beta}} J \succeq 0$
Gram $\succeq 0$ rank(Gram) $\le d$.	

(RANK): $\operatorname{rank}(\lambda I - A_G + \mu J) \le d$.

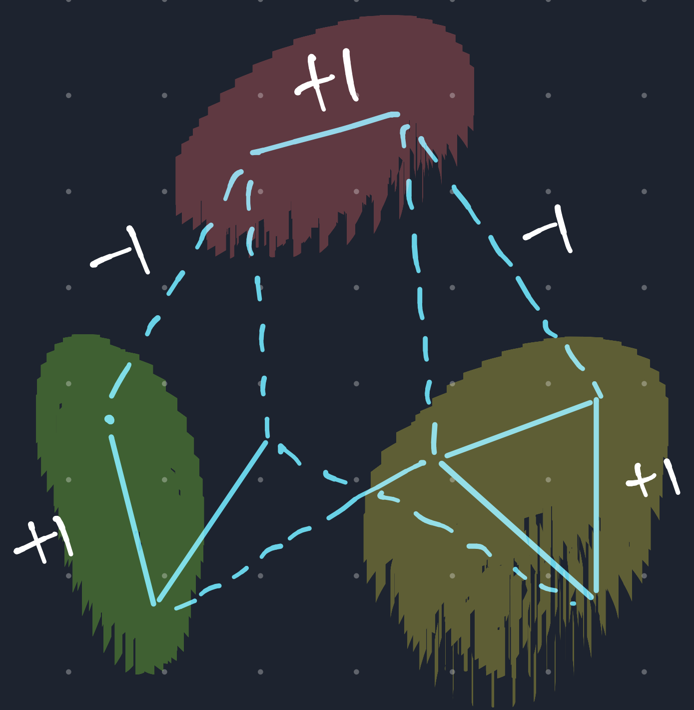
Goal: Given λ, μ, d, find largest N-vtx graph G with (PSD) & (RANK).

Goal Given λ, μ, d find largest N-vertex G s.t. $\lambda I - A_G + \mu J \succeq 0$

and $\operatorname{rank}(\lambda I - A_G + \mu J) \leq d$.

DEF Sign graph $G^{\pm}$.

Adj mat $\begin{bmatrix} 0 & & \pm 1 \\ & \ddots & \\ \pm 1 & & 0 \\ & & & 0 \end{bmatrix}$



$\lambda_1(G^{\pm})$ = largest e.v. of adj matrix.

A valid t-coloring of $G^{\pm}$, is a coloring of vertices using t-colors. s.t. + edge are colored the same and − edge are colored diffly.

$\chi(G^{\pm})$ = Smallest t s.t. $\exists$ valid t-coloring of $G^{\pm}$.

Construction

Question Generalize $k(\lambda)$ for $N_{\alpha,\beta}(d)$

$$\lambda = \frac{1-\alpha}{\alpha-\beta}$$

$$\mu = \frac{\alpha}{\alpha-\beta}$$

$$p = \lfloor 1/\mu \rfloor.$$

Def $$k_p(\lambda) = \inf\left\{ \frac{|G^{\pm}|}{\mathrm{mult}(\lambda, G^{\pm})} : \begin{array}{l} \lambda_1(G^{\pm}) = \lambda \\ \chi(G^{\pm}) \le p \end{array} \right\}.$$

PROP $$N_{\alpha,\beta}(d) \geq \frac{k_p(\lambda)\, d}{k_p(\lambda) - 1} + o(d).$$

CONJ The reverse inequality holds

$\underline{Q}$: $\lambda = 2.$

Examples

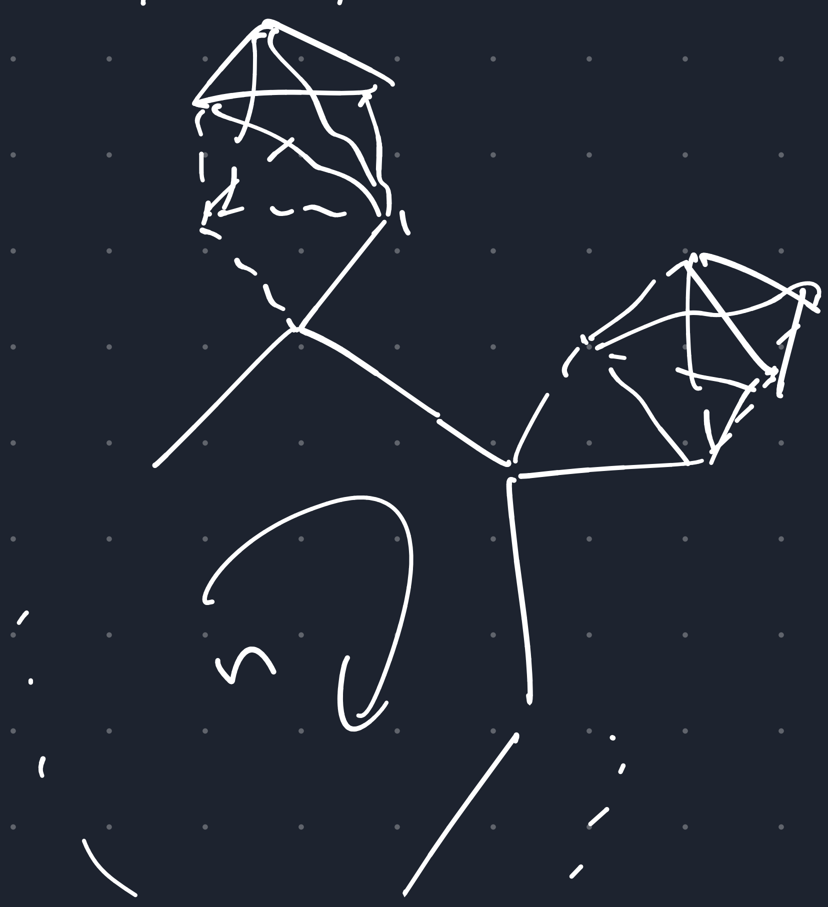
	α	β	λ	p	$k_p(\lambda)$	$N_{\alpha,\beta}(d)$
equiangular	α	$-\alpha$	$\frac{1-\alpha}{2\alpha}$	$2.$	$k(\lambda)$	$\frac{k(\lambda)}{k(\lambda)-1} d.$
- ish.	$\alpha + 2\beta < 0$		$\frac{1-\alpha}{\alpha-\beta}$	$\le 2.$	$k(\lambda)$	$\frac{k(\lambda)}{k(\lambda)-1} d.$
			1	$\ge 3.$	$p.$	$\frac{p}{p-1} d.$
			$\sqrt{2}$	≥ 3	2	$2d$
			$\sqrt{3}$	3	$7/3$	$7/4\, d$
			$\sqrt{3}$	≥ 4	2	$2d.$

Main tool $\forall j = 2$ and Δ $\exists C$ s.t. $\forall$ connected G with max degree $\leq \Delta$. jth eigenvalue multiplicity $\leq \dfrac{Cn}{\log\log n}$.

$o(n)$.

① Sperical 2-dist. generalizes to signed graphs only when $\chi(G^{\pm}) \leq 2$.

② Fails $\chi(G^{\pm}) \geq 3$



$|G^{\pm}| = 6n$

$\text{mult}(\lambda_1(G_n^{\pm}), G_n^{\pm}) = n.$