Spherical two-distance sets and spectral theory of signed graphs

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Joint work with Tidor, Yao, Zhang and Zhao (MIT team)

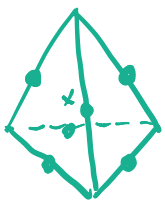
Spherical 2-dist set $= \{ v_1, \cdots, v_N \in \mathbb{R}^d : |v_i| = 1,$
$\langle v_i, v_j \rangle$ takes only 2 values $\forall i \neq j \}$.

Question: Find maximum size of spherical 2-dist set in $\mathbb{R}^d$.

$$\frac{1}{2} d(d+1) = \binom{d+1}{2} \leq N(d) \leq \frac{1}{2} d(d+3)$$

Delsarte, Goethals, Seidel

Example:



regular tetrahedron

$6 = \binom{3+1}{2}$.

[Glazyrin, Yu]: $N(d) = \frac{1}{2} d(d+1)$ whenever $d \geq 7$ and $d \neq (2k+1)^2 - 3$

Question: What if the inner products are fixed?

Given $-1 \leq \beta < \alpha < 1$.

$$\boxed{N(d) = \max_{\beta < \alpha} N_{\alpha,\beta}(d).}$$

$N_{\alpha,\beta}(d)$ = max size of set of unit vectors $v_1, \cdots, v_N$ in $\mathbb{R}^d$ s.t. $\langle v_i, v_j \rangle = \alpha$ or β $\forall i \neq j$.

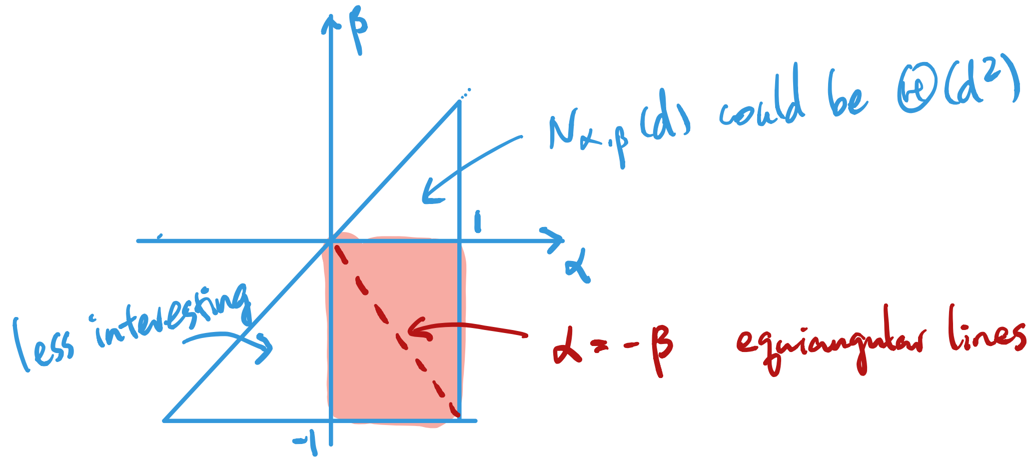
Review:

① [Neumair]. $N_{\alpha,\beta}(d) \leq 2d+1$ unless $\frac{1-\alpha}{\alpha-\beta} \in \mathbb{Z}$.

② [Larman, Rogers, Seidel].

$N_{\alpha,\beta}(d) = \Theta(d^2)$. if $0 \le \beta < \alpha < 1$. and $\frac{1-\alpha}{\alpha-\beta} \in \mathbb{Z}$.

③ $N_{\alpha,\beta}(d) \le 1 - \frac{1}{\alpha}$ if $-1 \le \beta < \alpha < 0$



④ [Balla, Dräxler, Keevash. Sudakov].

$$N_{\alpha,\beta}(d) \le 2\left(1 - \frac{\alpha}{\beta}\right) d + o(d) \quad \text{if} \quad -1 \le \beta < 0 \le \alpha < 1.$$

Problem: Fix $-1 \le \beta < 0 \le \alpha < 1$. Determine $N_{\alpha,\beta}(d)$ for large d.

In particular, find $\lim_{d\to\infty} \frac{N_{\alpha,\beta}(d)}{d}$

Equiangular line $\alpha = -\beta$

DEF: Spectral radius order $k(\lambda)$ = smallest k. s.t.

$\exists$ k-vertex graph G with $\lambda_1(G) = \lambda$.

↑ largest eigenvalue of adj mat. of G.

Set $k(\lambda) = \infty$ if no such G exists.

THM [JTYYZ 19+] Fix $\alpha > 0$, set $\lambda = \frac{1-\alpha}{2\alpha}$. for $d \ge d_0(\alpha)$

$$N_{\alpha,-\alpha}(d) = \begin{cases} \left\lfloor \frac{k(\lambda)(d-1)}{k(\lambda)-1} \right\rfloor. & \text{if } k(\lambda) < \infty \\ d + o(d) & \text{o/w.} \end{cases}$$

Example:

α	λ	G	$k(\lambda)$	$N_{\alpha,-\alpha}(d) \approx$
$1/3$	1		2	$2d$
$1/(1+2\sqrt{2})$	$\sqrt{2}$		3	$\frac{3d}{2}$
$1/5$	2		3	$\frac{3d}{2}$

Connection between $N_{\alpha,-\alpha}(d)$ and $k(\lambda)$

Equiangular lines in $\mathbb{R}^d$

$V = \{v_1, v_2, \dots, v_N\} \subseteq \mathbb{R}^d$.

$|v_i| = 1$ and $\langle v_i, v_j \rangle = \pm\alpha$

(*) $\begin{cases} \text{Gram matrix } (\langle v_i, v_j \rangle)_{ij} \succeq 0 \\ \text{rank}(\text{Gram mat}) \le d. \end{cases}$

$\approx$

N-vertex Graph G

$V = \{v_1, \dots, v_N\}$.

$v_i \sim v_j$ iff $\langle v_i, v_j \rangle = -\alpha$.

(PSD): $\lambda I - A_G + \frac{1}{2}J \succeq 0$

$\lambda = \frac{1-\alpha}{2\alpha}$ adj. mat. all-ones matrix

(RANK): $\text{rank}(\lambda I - A_G + \frac{1}{2}J) \le d$

Goal: Given d. find largest N s.t. $\exists$ N-vertex graph G with (PSD) + (RANK).

Example: Given $\lambda = \frac{1-\alpha}{2\alpha}$. find k-vertex graph G_0 with $\lambda_1(G_0) = \lambda$

Take G = disjoint ℓ copies of G_0 (ℓ to be determined).

Check (PSD): $\{\text{eigenvalues of } G\} = \{\underbrace{\lambda, \lambda, \dots, \lambda}_{\ell}, \dots\}$

$$\underbrace{\lambda I - A_G}_{\succeq 0} + \underbrace{\tfrac{1}{2}J}_{\succeq 0} \succeq 0.$$

Take $\ell \approx \frac{d}{k-1}$

rank 1

(RANK): $\text{rank}(\lambda I - A_G + \frac{1}{2}J) \le \ell(k-1) + 1 \le d$

$|G| = \ell \cdot k \approx \frac{d}{k-1} \cdot k \approx \frac{k}{k-1} d.$ Want k small

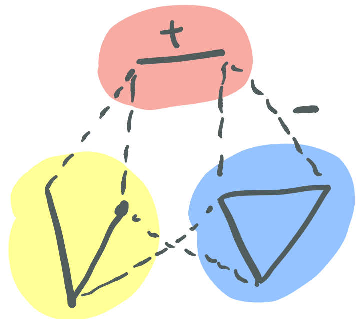
Better take $k = k(\lambda)$

COR: $N_{\alpha,-\alpha}(d) \geq \dfrac{k(\alpha)\, d}{k(\alpha)-1} + o(1)$.

THM [JTYYZ 19+]. The reversed inequality holds (HARD).

Question: Is there an analog of $k(\alpha)$ for $N_{\alpha,\beta}(d)$?

DEF: Signed graph $G^{\pm}$



Adj mat $\begin{bmatrix} 0 & & \pm 1 \\ & \ddots & \\ \pm 1 & & 0 \end{bmatrix}$

$\lambda_1(G^{\pm})$ = largest e.v. of $A_{G^{\pm}}$.

A valid t-coloring of $G^{\pm}$ is a coloring of the vertices using t-colors s.t. endpoints of every + edge are colored the same and - - - - - - - - - - are colored differently.

Chromatic number of $G^{\pm}$: $\chi(G^{\pm})$ = smallest t s.t. $\exists$ valid t-color.

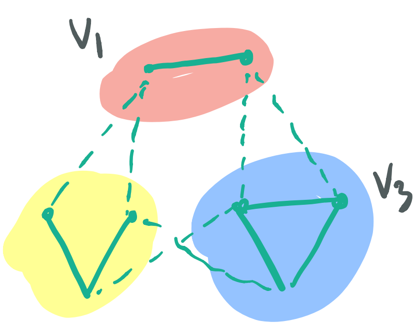
Example: Fix $-1 \leq \beta < 0 \leq \alpha < 1$. Set $\lambda = \dfrac{1-\alpha}{\alpha-\beta}$, $p = \left\lfloor \dfrac{-\alpha}{\beta} \right\rfloor + 1$, $\mu = \dfrac{\alpha}{\alpha-\beta}$.

Take $G^{\pm}$ with $\chi(G^{\pm}) \leq p$, and $\lambda_1(G^{\pm}) = \lambda$.

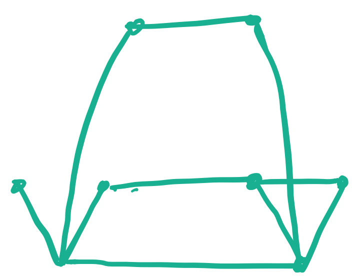
For $v_i \in V_i$, $v_j \in V_j$ $(i \neq j)$

$v_i v_j$ = − edge in $G^{\pm}$ $\mapsto$ non-edge in G

$v_i v_j$ = nonedge in $G^{\pm}$ $\mapsto$ edge in G



$G^{\pm}$



$= G$.

(PSD) ✓ Can be checked.

(RANK) $\operatorname{rank}(\lambda I - A_G + \mu J) = \operatorname{rank}((\lambda I - A_{G^\pm}) + (A_{G^\pm} - A_G + \mu J))$

$$\le |G^\pm| - \operatorname{mult}(\lambda, A_{G^\pm}) + p \underset{\text{want}}{\le} d.$$

Spherical 2-dist set	N-vertex graph G.
$\{v_1, \dots, v_N\} \subseteq \mathbb{R}^d$.	$v_i \sim v_j$ iff $\langle v_i, v_j \rangle = \beta$.
$\|v_i\| = 1$. $\langle v_i, v_j \rangle = \alpha$ or β. $\approx$	(PSD): $\underset{\uparrow\ \frac{1-\alpha}{\alpha-\beta}}{\lambda} I - A_G + \underset{\uparrow\ \frac{\alpha}{\alpha-\beta}}{\mu} J \succeq 0$
Gram matrix $\succeq 0$	
$\operatorname{rank}(\text{Gram}) \le d$	(RANK): $\operatorname{rank}(\lambda I - A_G + \mu J) \le d$.

Goal: Given d, find largest N s.t. $\exists$ N-vtx graph G with (PSD) + (RANK)

Want $|G^\pm| - \operatorname{mult}(\lambda, A_{G^\pm}) + \underline{p} \le d$

$$\Rightarrow \underline{\underline{|G^\pm|}} \ge \frac{d}{1 - \underline{\underline{\operatorname{mult}(\lambda, A_{G^\pm}) / |G^\pm|}}} + O(1)$$

Want $\max \dfrac{\operatorname{mult}(\lambda, A_{G^\pm})}{|G^\pm|}$.

Def: $k_p(\lambda) = \inf \left\{ \dfrac{|G^\pm|}{\operatorname{mult}(\lambda, A_{G^\pm})} : G^\pm \text{ satisfies } \begin{matrix} \chi(G^\pm) \le p \\ \lambda_1(G^\pm) = \lambda \end{matrix} \right\}$

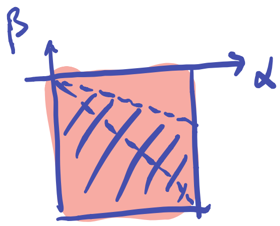
COR: Fix $-1 \le \beta < 0 \le \alpha < 1$. set $\lambda = \frac{1-\alpha}{\alpha - \beta}$. $p = \lfloor \frac{-\alpha}{\beta} \rfloor + 1$.

Then $N_{\alpha,\beta}(d) \ge \dfrac{k_p(\lambda)}{k_p(\lambda) - 1} d + o(d)$.

CONJ: The reversed inequality holds.

Example:

	α	β	λ	p	$k_p(\lambda)$	$N_{\alpha,p}(d)$
Equiangular line-ish	α	$-\alpha$	$\frac{1-\alpha}{2\alpha}$	2	$k(\lambda)$	$\frac{k(\lambda)}{k(\lambda)-1}d$
	$\alpha+2\beta<0$		$\frac{1-\alpha}{\alpha-\beta}$	≤ 2	$k(\lambda)$	$\frac{k(\lambda)}{k(\lambda)-1}d$
$\sum \lambda_i^3(G^\pm)$			1	≥ 3	p	$\frac{p}{p-1}d$
algebraic method			$\sqrt{2}$	≥ 3	2	$2d$
			$\sqrt{3}$	3	$\frac{7}{3}$	$\frac{7}{4}d$
			$\sqrt{3}$	≥ 4	2	$2d$



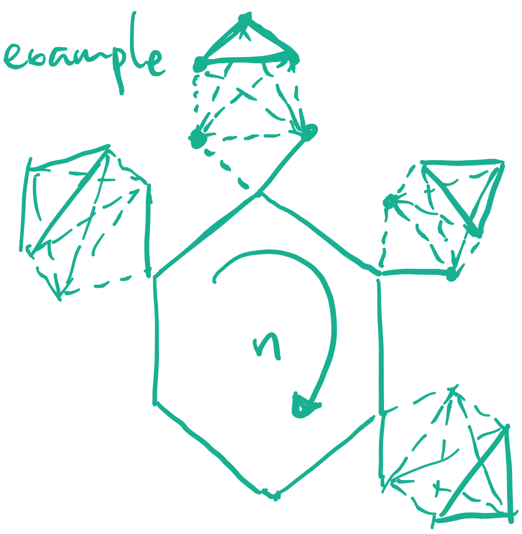
Equiangular line (-ish) $\alpha+2\beta<0$

(Main tool). $\forall j$ and Δ. $\exists C=C(\Delta,j)$ s.t. $\forall$ connected G with max deg $\leq \Delta$. $\operatorname{mult}(\lambda_j(G), A_G) \leq \frac{Cn}{\log\log n}$

① This tool generalizes to $G^\pm$ with $\chi(G^\pm)\leq 2$.

② This tool fails for $G^\pm$ as soon as $\chi(G^\pm)\geq 3$.

Counterexample



$G_n^\pm$

$\operatorname{mult}(\lambda_1(G_n^\pm), A_{G_n^\pm}) = n.$

$|G_n^\pm| = 6n.$