Riddle (RMM 2020. Day 1. Problem 3).

Let $n \geq 3$ be an integer. In a country there are n airports and n airlines operating two-way flights. For each airline, there is an odd integer $m \geq 3$, and m distinct airports $c_1, \ldots, c_m$, where the flights offered by the airline are exactly those between the following pairs of airports: c_1 and c_2; c_2 and c_3; $\ldots$; c_{m-1} and c_m; c_m and c_1.

Prove that there is a closed route consisting of an odd number of flights where no two flights are operated by the same airline.

Rainbow odd cycles (and other short stories)

joint work with Ron Aharoni, Ron Holzman & Joseph Briggs

<u>DEF</u>: Given a family (multiset) $\mathcal{E}$ of sets, an <u>$\mathcal{E}$-rainbow set</u> $R \subseteq \bigcup \mathcal{E}$ with an injection $\sigma: R \to \mathcal{E}$ s.t. $\sigma(e) \ni e \ \forall e \in R$.

<u>Problem</u>: Given a property $\mathcal{P}$, find smallest $m = m(\mathcal{P})$ s.t. the following holds: For every family $\mathcal{E}$, if $|\mathcal{E}| \geq m$ and every member of $\mathcal{E}$ satisfies $\mathcal{P}$, then $\exists$ a $\mathcal{E}$-rainbow set R with $\mathcal{P}$.

<u>Examples</u> ① Colorful Carathéodory theorem (Bárány '82).

Every family of <u>$n+1$</u> subsets of $\mathbb{R}^n$, each containing $\vec{a}$

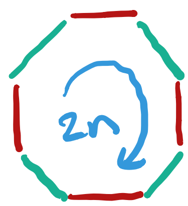
in its convex hull, has a rainbow set with the same property.

②. (Drisko '98, Aharoni–Berger '09).

$2n-1$ matchings of size n in any bipartite graph have a rainbow matching of size n.

Remark: ① n subsets of $\mathbb{R}^n$ each $\ni \vec{a}$ in its conv. hull, generically do NOT have a rainbow set with the same property.

②. (Aharoni, Kotlar, Ziv '18) $2n-2$ matchings of size n in any bipartite graph DO have a rainbow matching of size n UNLESS



$\times (n-1)$

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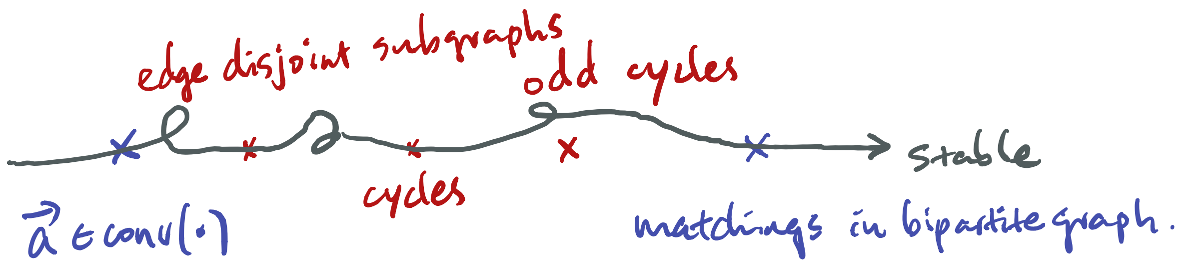
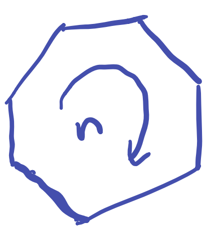


Riddle: Every family of n odd cycles in K_n has a rainbow odd cycle.

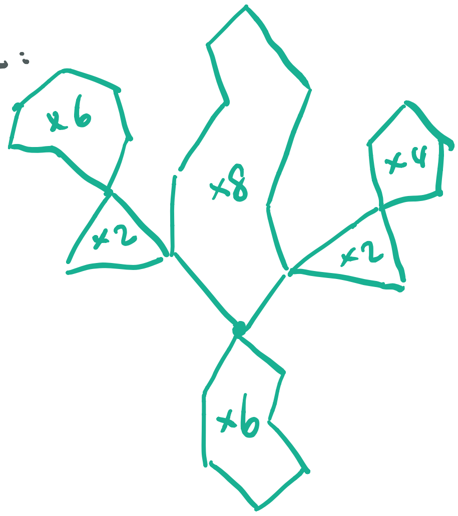
Remark:



$\times (n-1)$ show sharpness of ↑.

(works only when n is odd).

DEF:



A family $\mathcal{O}$ of cycles is a __pruned cactus__ if all the cycles are identical to a cycle on $|\mathcal{O}|+1$ vertices

__OR__ $\mathcal{O}$ can be partitioned into 2 pruned cacti $\mathcal{O}_1, \mathcal{O}_2$ s.t. $\cup\mathcal{O}_1$, $\cup\mathcal{O}_2$ share exactly one vertex

__THM:__ If a family $\mathcal{O}$ of $\cancel{n-1}$ n odd cycles in $\cancel{K_n}$ K_{n+1} has no rainbow odd cycle, then it is a pruned cactus.

__COR:__ When n is even, every family of $n-1$ odd cycles in K_n (cannot be a pruned cactus) has a rainbow odd cycle.

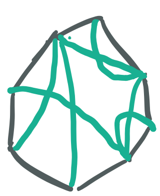
__Proof Sketch of THM:__ Break into 3 cases. Suppose $|V(\cup\mathcal{O})| \leq |\mathcal{O}|+1$.

__Case 1:__ There exists $\mathcal{K} \subsetneq \mathcal{O}$ s.t. $|V(\cup\mathcal{K})| \leq |\mathcal{K}|+1$.

__Case 2:__ Every odd cycle in $\mathcal{O}$ is Hamiltonian.

__Case 3:__ For every $\mathcal{K} \subsetneq \mathcal{O}$, $|V(\cup\mathcal{K})| > |\mathcal{K}|+1$, and some $O_i \in \mathcal{O}$ is not Hamiltonian.

__proof of Case 3:__ $\mathcal{O} = \{O_1, O_2, \cdots, O_n\}$. Say O_n is not Ham.

O_n  $\cdot v$. $V = V(K_{n+1}) \setminus \{v\}$. $/\!/ \mathcal{O}'$.

Consider $\{O_1[v], O_2[v], \ldots, O_{n-1}[v]\}$.

They are connected subgraphs s.t. $\forall\, \mathcal{K}' \subseteq \Theta'$.

$$|V(\cup \mathcal{K}')| \geq |\mathcal{K}'| + 1$$

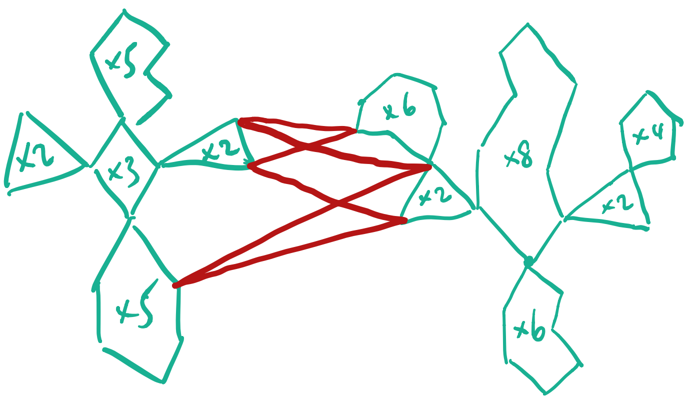
Rado's matroid thm $\Longrightarrow$ Θ' has a rainbow tree spanning V.

$\xRightarrow{\text{Using On}}$ rainbow odd cycle. ∎

Rainbow cycles:

Prop: Every family of n cycles in K_n has a rainbow cycle.

Remark: Pruned cacti give sharpness ↑.

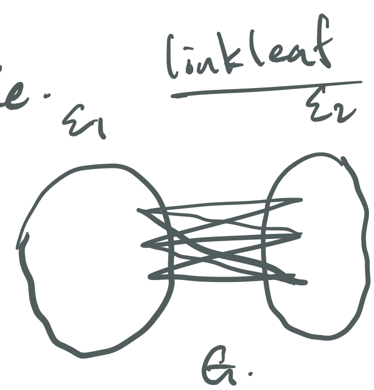


recursively $\Longrightarrow$ saguaro

Thm: For every family Θ of $n-1$ cycles in K_n, no rainbow cycle exists iff Θ is a saguaro.

Edge-disjoint subgraphs

Prop: Every family of n edge-disjoint non-empty subgraphs in K_n has a rainbow cycle.



Remark: $\mathcal{E}. = \underline{\underline{\mathcal{E}_1}} \cup \{G\} \cup \underline{\underline{\mathcal{E}_2}}.$

Thm: For Every family of $n-1$ edge disjoint nonempty subgraphs in K_n. no rainbow cycles exists $\Leftrightarrow$ the family is linkleaf.

Even cycles

cannot improve it beyond $\frac{6}{5}n$.

Prop: Every family of $\left\lfloor \frac{3(n-1)}{2} \right\rfloor + 1$ even cycles in K_n has a rainbow even cycle.