# Equiangular lines with a fixed angle

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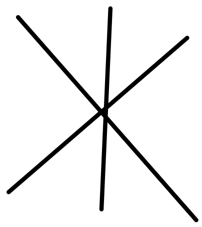
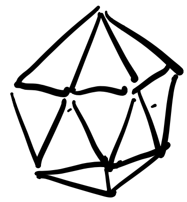
arxiv: 1708.02317 arxiv: 1907.12466

Lines in $\mathbb{R}^n$ (through O) pairwise separated by same angle

$\mathbb{R}^2$:  3 lines

$\mathbb{R}^3$:  6 lines

**Question:** maximum size of equi. lines in $\mathbb{R}^n$?

| n | 2 | 3-4 | 5 | 6 | 7-14 | ... | 23-41 | 42 | ... |
|---|---|---|---|---|---|---|---|---|---|
| max | 3 | 6 | 10 | 16 | 28 | ... | 276 | 276-288 | ... |

$$cn^2 \leq \max \leq \binom{n+1}{2}$$

de Caen 2000 Gerzon 1973

* * *

**Question:** What if the angle is fixed?

$E_\alpha(n)$ = max size of equiangular lines with angle $\arccos\alpha$ in $\mathbb{R}^n$.

1973. Lemmens–Seidel $E_{1/3}(n) = 2(n-1)$ for $n \geq 15$

1989 Neumaier $E_{1/5}(n) = \lfloor \frac{3}{2}(n-1) \rfloor$ for $n \gg 1$.

1973 Neumann $E_\alpha(n) \leq 2n$ unless $1/\alpha$ is odd.

2016 Bukh $E_\alpha(n) \leq C_\alpha n$. ($C_\alpha$ depends on $\alpha$)

2018 Balla, Dräxler, Sudakov, Keevash $E_\alpha(n) \leq 1.93n$ if $n \geq n_0(\alpha)$ and $\alpha \neq 1/3$

**Conjecture 1** (Bukh): $E_{1/7}(n) \approx \frac{4}{3}n$ $\quad E_{\frac{1}{2k-1}}(n) \approx \frac{k}{k-1}n.$

**Conjecture 2** (J.-Polyanskii)

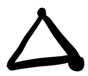
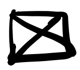
$$E_\alpha(n) \approx \frac{k}{k-1}n, \quad \text{where } k = k(\lambda), \ \lambda = \frac{1-\alpha}{2\alpha}$$

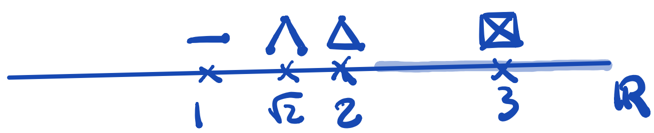
**Spectral radius order**

$k(\lambda) :=$ smallest $k$ s.t. $\exists$ $k$-vertex graph $G$ s.t. $\lambda_1(G) = \lambda$.

$\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_k(G)$

the eigenvalues of adjacency matrix

| $\alpha$ | $\lambda$ | $G$ | $k$ | $E_\alpha(n)$ |
|---|---|---|---|---|
| $1/3$ | $1$ |  | $2$ | $2n$ |
| $1/5$ | $2$ |  | $3$ | $\frac{3}{2}n$ |
| $1/7$ | $3$ |  | $4$ | $\frac{4}{3}n$ |




**THM** (J.-Polyanskii)

Conj 2 holds for all $\lambda \leq \sqrt{2+\sqrt{5}}$.

Barrier

Spectral radii of graphs is dense in $(\sqrt{2+\sqrt{5}}, \infty)$.

**THM** (JTYZZ)

$$E_\alpha(n) = \left\lfloor \frac{k}{k-1}(n-1) \right\rfloor \quad \text{when } k(\lambda) < \infty \qquad n \geq n_0(\alpha)$$

$$E_\alpha(n) = n + o(n) \quad \text{when } k(\lambda) = \infty$$

**Remark** When $\alpha = \frac{1}{2k-1}$, can show $k(\lambda) = k$,

hence $E_\alpha(n) = \left\lfloor \frac{k}{k-1}(n-1) \right\rfloor$, $n \geq n_0(\alpha)$

Equiangular lines in $\mathbb{R}^n$

$V :=$ set of unit vectors

(each vector represents a line)

$\langle v_1, v_2 \rangle = \pm \alpha$

Gram matrix $(\langle v_i, v_j \rangle)_{i,j} \succeq 0$

$\operatorname{rank}(\text{Gram mat.}) \le n$

$m$-vertex graph $G$.

$V$ vertex set

(PSD): $\lambda I - A + \frac{1}{2} J \succeq 0$

$\frac{1-\alpha}{2\alpha}$ adj mat. all ones matrix

(RANK): $\operatorname{rank}(\lambda I - A + \frac{1}{2} J) \le n$

Think as if

$\operatorname{rank}(\lambda I - A) \le n$

<u>Goal</u>: Given $n$. find largest $m$

s.t. an $m$-vtx graph $G$ with (PSD) + (RANK).

<u>Alternative goal</u>: Given $m$. find smallest $n$ s.t.

an $m$-vtx graph $G$ s.t. (PSD) + (RANK).

In other words, given $m$. minimize $\operatorname{rank}(\lambda I - A)$

$\Leftrightarrow$ maximize $\operatorname{mult}(\lambda, G)$

(PSD) $\Longrightarrow$ We need to deal with 2 cases

(Completely reducible) $G = G_1 \cup \cdots \cup G_c$ where

each connected component $G_i$ satisfies $\lambda_1(G_i) = \lambda$

$\operatorname{mult}(\lambda, G) = \sum \operatorname{mult}(\lambda, G_i) = c \approx \frac{m}{k(\lambda)}$

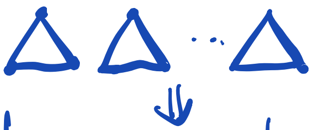
In this case, it is optimal to take $|G_i| = k(\lambda)$.

(Irreducible) : $G$ is connected where $\lambda_2(G) = \lambda$.

$$\mathrm{mult}(\lambda, G) = o(m)$$

<u>THM</u> (JTYZZ). Given an $n$-vertex connected graph $G$ with max deg of $G \le \Delta$. If $\lambda$ is $\lambda_2(G)$. then

$$\mathrm{mult}(\lambda, G) \le \frac{n}{\log\log n} = o(n).$$

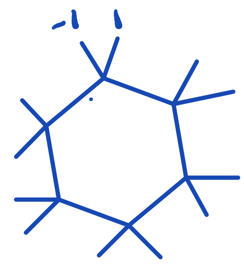


need: Connectedness.

SRG.

$\Downarrow$

max deg $\le \Delta$



$\lambda = \lambda_2(G)$

<u>Question</u> Is it true that $\mathrm{mult}(\lambda, G) \le n^{1-\varepsilon}$. ?