Beyond classification theorem of Cameron, Goethals, Seidel and Shult (CGSS).

<u>Fundamental problem</u>: Characterization of graphs with limited eigenvalues.

$$\mathcal{G}(\lambda) = \{ \text{graphs } G \text{ with smallest eigenvalue } \lambda_1(G) \geq -\lambda \}$$

(refer to adjacency matrix).

- Review of $\mathcal{G}(2)$
- Classification theorem of $\mathcal{G}(\lambda^*) \setminus \mathcal{G}(2)$.
- Beyond $\mathcal{G}(\lambda^*)$

<u>Review of $\mathcal{G}(2)$</u>:

Well known: $\mathcal{G}(2) \supseteq \{\text{line graphs}\}$.

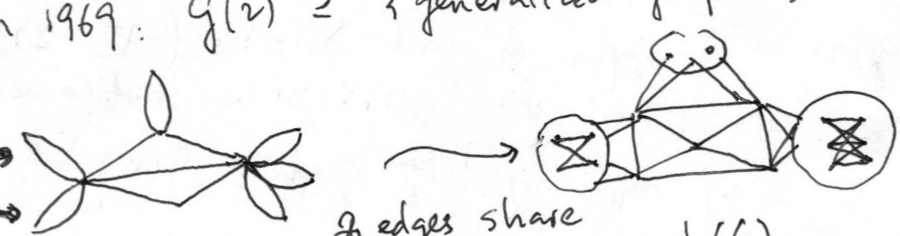
no 

G $\qquad\qquad$ $L(G)$

edge-vertex incidence matrix B: $\qquad$ adjacency matrix $A = BB^T - 2I$.

Hoffman 1969: $\mathcal{G}(2) \supseteq \{\text{generalized line graphs}\}$.

petals



G $\qquad$ 2 edges share exactly one vtx $\qquad$ $L(G)$

... Note $\mathcal{G}(\lambda)$ is closed under taking disjoint union.

<u>CGSS</u>:

(1976)

$G \in \mathcal{G}(2) \leftrightarrow \frac{1}{2}A_G + I \succeq 0 \leftrightarrow$ unit vectors.

$\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 1 & & 1 \end{pmatrix}$ pairwise angle 60° or 90°

representation theory of semisimple Lie algebra.

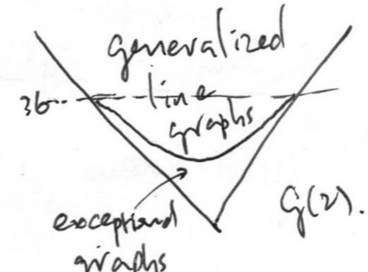
<u>Classification thm</u>:

$\{$connected graphs in $\mathcal{G}(2)\}$ = $\{$generalized line graphs$\}$

$\cup$ $\{$exceptional graphs$\}$

represented by subset of E_8 root system.

at most 36 vertices



- Classification theorem of $\mathcal{G}(\lambda^*) \setminus \mathcal{G}(2)$.

$\lambda^* = 2.019800887\ldots$

$\lambda_1($  $E_n) \searrow -\lambda^*$.

$|\leftarrow n \rightarrow|$

λ^* not totally real. cannot be graph e.v.

(some conjugates are not real)

$\mathcal{G}(\lambda^*) \setminus \mathcal{G}(2) = \{$graphs G with $\lambda_1(G) \in (-\lambda^*, -2)\}$

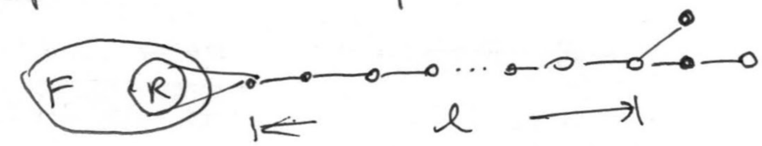
<u>THM</u>: $\forall \lambda \in (2, \lambda^*)$. $\mathcal{G}(\lambda) \setminus \mathcal{G}(2)$ has finitely connected graphs

" Every sufficiently large graph in $\mathcal{G}(\lambda^*) \setminus \mathcal{G}(2)$

looks more or less like E_n ".

DEF: A rooted graph F_R is a graph F equipped with nonempty subset R of vertices

The augmented path extension (ape) $(F_R, \ell, \cdot)$ of F_R is defined by



Part 1: Every sufficiently large connected $G \in \mathcal{G}(\lambda^*) \setminus \mathcal{G}(2)$ is an ape of a <u>rooted graph</u>.

↑ classify this ?

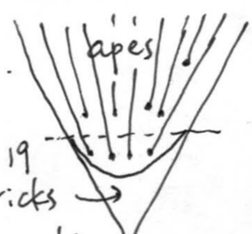
DEF: A single-rooted graph H_r is a rooted graph H with a single root r. The line graph $L(H_r)$ is the rooted graph F_R where $F = L(H)$ and R $\{$edges of H incident to $r\}$

Part 2: There exists a finite family $\mathcal{F}$ of rooted graphs s.t.

① every F_R in $\mathcal{F}$ is $L(H_r)$ for some connected bipartite single rooted H_r.

② every connected ape in $\mathcal{G}(\lambda^*) \setminus \mathcal{G}(2)$ is an ape of some F_R in $\mathcal{F}$.

③ for every F_R in $\mathcal{F}$, there exists $\ell_0 \in \{0, \dots, 6\}$ s.t. $\lambda_1(F_R, \ell, \cdot) \in (-\lambda^*, -2)$ iff $\ell \geq \ell_0$.



Quantative version: enum 794 of H_r. (794 total)

(48 maximal)

computer assisted.

DEF. A <u>maverick graph</u> is a connected graph in $\mathcal{G}(\lambda^*) \setminus \mathcal{G}(2)$ that is not an ape.

Part 3: Enumeration of 4752 maverick graphs.

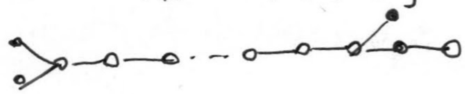
at most 19 vertices. (computer assisted).

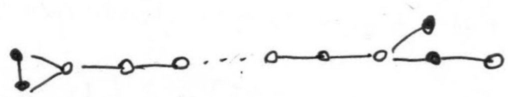
order	17	18	19
#	42	13	3

Key linear algebraic lemma: (mention after Part 2).

$$(F_R, l_1, \cdot) \in \mathcal{G}(\lambda^*) \Leftrightarrow (F_R, 0, \cdot) \in \mathcal{G}(\lambda^*).$$

Example: Consider two cases $F_R \in \{ \cdot, \cdot \}$

$(\,:\,, l_1, \cdot) =$  $\lambda_1 \searrow -\lambda^*$

$(\,\cdot\,, l, \cdot) =$  $\lambda_1 \nearrow -\lambda^*$

COR: If $\lambda_1(F_R, l, \cdot) > -\lambda^*$, then $F[R]$ is complete.

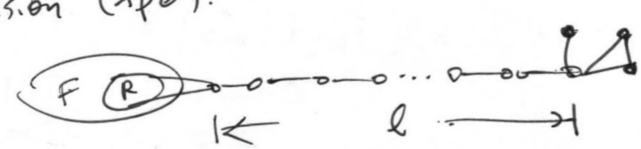
Pf: ....

app

Beyond $\mathcal{G}(\lambda^*)$.

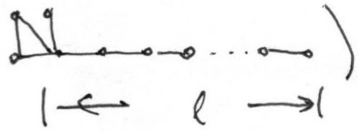
A notable portion of maverick graphs look alike

DFE: Twisted path extension (tpe).

$(F_R, l, \cdot) =$ 

Thm: There're 1161 twisted maverick graphs.

order	17	18	19
#	40	13	3

$\lambda_1\Big($  $\Big) \downarrow -\lambda'$, $\lambda' \approx 2.02124$.

<u>Generalization of Part 1.</u>

$\forall \lambda \in (\lambda^*, \lambda')$ every sufficiently large graph in $\mathcal{G}(\lambda) \setminus \mathcal{G}(2)$ is an ape.

<u>Nogo thm for Part 2</u>: $\forall \lambda > \lambda^*$, finite $\mathcal{F}$ of rooted graphs, $N \in \mathbb{N}$. $\exists G$ on $> N$ vertices in $\mathcal{G}(\lambda) \setminus \mathcal{G}(2)$ that is not an ape of $F \in \mathcal{F}$.

<u>~~App~~</u>: Every connected graph G on ≥ 18 vertices in $\mathcal{G}(\lambda^*) \setminus \mathcal{G}(2)$ contains a unique leaf u s.t. $G - u$ is the line graph of a bipartite graph.